



1.6.8.5 The complex conjugate

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Introduction

- In this topic, we will
 - Define the conjugate of a complex number
 - Look at both rectangular and polar representations
 - Look at a geometric interpretation
 - Observe some properties





Review

- Recall from secondary school that when you rationalized the denominator of an expression, such as

$$\frac{1}{1+2\sqrt{3}} = \frac{1}{1+2\sqrt{3}} \cdot \frac{1-2\sqrt{3}}{1-2\sqrt{3}} = \frac{1-2\sqrt{3}}{1-12} = \frac{2\sqrt{3}-1}{11}$$

- You referred to $1-2\sqrt{3}$ as the *conjugate* or *radical conjugate* of the denominator $1+2\sqrt{3}$





The complex conjugate

- Recall that $j \stackrel{\text{def}}{=} \sqrt{-1}$ and so, given $z = \alpha + \beta j$, it is reasonable to define the *complex conjugate* of z as

$$z^* = \alpha - \beta j$$

- One common use of the complex conjugate is to do exactly that:
 - Rationalize the denominator

$$\frac{1+j}{2-j} = \frac{1+j}{2-j} \cdot \frac{2+j}{2+j} = \frac{1+3j}{5} = 0.2 + 0.6j$$

- This converts the ratio of complex numbers into the rectangular representation





The complex conjugate

- Some examples:

$$(3 + 4j)^* = 3 - 4j$$

$$(-2 + j)^* = -2 - j$$

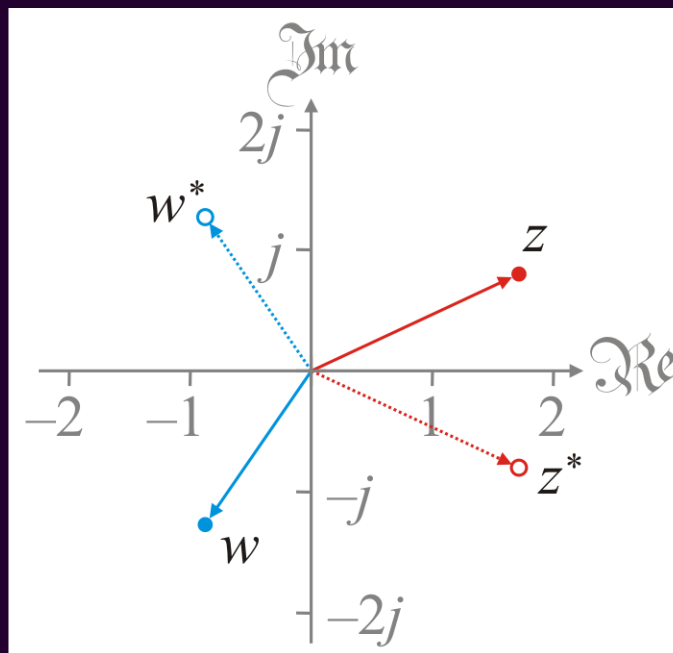
$$(5 - 7j)^* = 5 + 7j$$





Geometric interpretation

- The geometric interpretation of the complex conjugate is a reflection through the real axis





Polar representation

- If a complex number is in the polar representation, the complex conjugate is equally easy to find:
 - If $z = |z| \angle \theta$ then $z^* = |z| \angle (-\theta)$
- To show this, just recall that

$$z = |z| \angle \theta = |z| \cos(\theta) + |z| \sin(\theta) j$$

so

$$\begin{aligned} |z| \angle (-\theta) &= |z| \cos(-\theta) + |z| \sin(-\theta) j \\ &= |z| \cos(\theta) - |z| \sin(\theta) j \\ &= z^* \end{aligned}$$





Polar representation

- There are many useful properties of the complex conjugate
 - The proofs may be done with either rectangular or polar representations
 - We'll use both





Properties

Theorem: $(z^*)^* = z$

- The conjugate of the conjugate of z is z

Proof: If $z = \alpha + \beta j$, then

$$\begin{aligned}(z^*)^* &= ((\alpha + \beta j)^*)^* \\ &= (\alpha - \beta j)^* \\ &= (\alpha - (-\beta j)) \\ &= \alpha + \beta j = z \quad \mathcal{QED}\end{aligned}$$





Properties

Theorem: $zz^* = |z|^2$

– z times the conjugate of z is the absolute value of z squared

Proof: If $z = |z| \angle \theta$, then

$$\begin{aligned} zz^* &= (|z| \angle \theta)(|z| \angle \theta)^* \\ &= (|z| \angle \theta)(|z| \angle (-\theta)) \\ &= |z|^2 \angle (\theta - \theta) \\ &= |z|^2 \angle 0 \\ &= |z|^2 \quad \text{QED} \end{aligned}$$





Properties

Theorem: $(z + w)^* = z^* + w^*$

- The conjugate of a sum is the sum of the conjugates

Proof: If $z = \alpha + \beta j$ and $w = \gamma + \delta j$,

$$\begin{aligned}(w + z)^* &= ((\alpha + \beta j) + (\gamma + \delta j))^* \\&= ((\alpha + \gamma) + (\beta + \delta)j)^* \\&= (\alpha + \gamma) - (\beta + \delta)j \\&= (\alpha - \beta j) + (\gamma - \delta j) = z^* + w^* \quad QED\end{aligned}$$





Properties

Theorem: $(zw)^* = z^* w^*$

- The conjugate of a product is the product of the conjugates

Proof: If $z = |z| \angle \theta$ and $w = |w| \angle \phi$, then

$$\begin{aligned}(zw)^* &= \left((|z| \angle \theta) (|w| \angle \phi) \right)^* \\&= \left(|z||w| \angle (\theta + \phi) \right)^* \\&= |z||w| \angle -(\theta + \phi) \\&= |z||w| \angle (-\theta - \phi) \\&= \left(|z| \angle (-\theta) \right) \left(|w| \angle (-\phi) \right) = z^* w^* \quad \text{QED}\end{aligned}$$





Properties

Theorem: $z = z^*$ if and only if z is real.

Proof: If $z = \alpha + \beta j$,

$$z = z^*$$

$$\Leftrightarrow \alpha + \beta j = (\alpha + \beta j)^*$$

$$\Leftrightarrow \alpha + \beta j = \alpha - \beta j$$

$$\Leftrightarrow \alpha = \alpha \text{ and } \beta = -\beta$$

$$\Leftrightarrow \alpha = \alpha \text{ and } \beta = 0$$

$$\Leftrightarrow z \text{ is real } \quad \mathcal{QED}$$

Theorem: $z = -z^*$ if and only if z is imaginary.





Properties

Theorem: $z + z^* = 2\Re(z)$ and $z - z^* = 2\Im(z)j$.

Proof: If $z = \alpha + \beta j$,

$$\begin{aligned} z + z^* &= (\alpha + \beta j) + (\alpha + \beta j)^* \\ &= \alpha + \beta j + \alpha - \beta j \\ &= 2\alpha = 2\Re(z) \end{aligned}$$

$$\begin{aligned} z - z^* &= (\alpha + \beta j) - (\alpha + \beta j)^* \\ &= \alpha + \beta j - \alpha + \beta j \\ &= 2\beta j = 2\Im(z)j \quad \text{QED} \end{aligned}$$





Properties

Theorem: $\Re(z) = \frac{z + z^*}{2}$ and $\Im(z) = \frac{z - z^*}{2j}$.

Proof: See the previous theorem.





Properties

Theorem: $-(z^*) = (-z)^*$.

Proof: If $z = \alpha + \beta j$, then

$$\begin{aligned} -(z^*) &= -(\alpha + \beta j)^* \\ &= -(\alpha - \beta j) \\ &= -\alpha + \beta j \\ &= (-\alpha - \beta j)^* \\ &= (-z)^* \quad \text{QED} \end{aligned}$$





Important takeaways

- The most important takeaways are
 - Complex conjugation is a reflection in the real axis
 - For most operations, it doesn't matter if you
 - Reflect first and then perform the operation, or
 - Perform the operation and then reflect





Summary

- In this topic, we've introduced the complex conjugate
 - It is a reflection in the real axis
 - Used to rationalize the denominator
 - We saw that
 - The conjugate of the conjugate of z is itself
 - For many operations, you can apply conjugation before or after the operation in question
 - The real and imaginary components can be extracted using conjugates





References

- [1] https://en.wikipedia.org/wiki/Complex_conjugate





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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